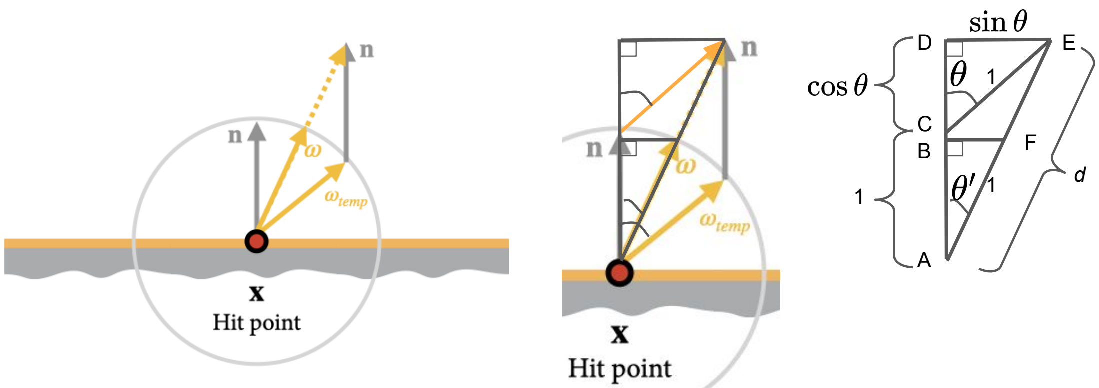
A quick proof for Pete Shirley's trick to sample cosine distributed directions without coordinate system transformation

In his rendering course[1], Prof. Derek Nowrouzezahrai presents a method for sampling cosine-distributed directions around the surface normal. We refer to this method as the Shirley Trick since it was apparently first introduced by Pete Shirley in his Graphics Blog[2]. The Shirley trick is very practical as it does not require to construct a local basis and transform coordinates from local surface basis to global world basis. We provide here proof that sampling directions according to the Shirley trick is indeed equivalent to sampling cosine-distributed directions around the normal.

The Shirley Trick

As described in Prof. Nowrouzezahrai's class:

There are many strategies for sampling cosine-distributed points (about the coordinate system defined by the surface normal $\mathbf{n}$ *at* $\mathbf{x}$*). Perhaps the simplest such approach is to first generate a uniform spherical sample (*$\boldsymbol{\omega}_{temp}$*), then add it to the surface normal, before finally normalizing the result as* $\boldsymbol{\omega}$*. The resulting direction will have cosine density aligned about the appropriate, aforementioned coordinate system.*



(a) (b)

Figure 1: (a) Illustration of the Sherley Trick for Sampling Directions with Cosine-density in the Shading Coordinate System. Excerpt from Prof. Nowrouzezahrai's class. (b) Defining distances, angles, and points from $\mathbf{n}, \boldsymbol{\omega}_{temp}$ and $\boldsymbol{\omega}$.

Proof. We follow Pharr et al. (2023) notations and results. Specifically from Section 13.5.3, for spherical coordinates applied to $\boldsymbol{\omega_{temp}}$ we have that $p(\theta,\phi) = \sin\theta p(\omega_{temp})$. Additionally, since $\boldsymbol{\omega_{temp}}$ are sampled uniformly on the unit sphere we have that $p(\omega_{temp}) = 1/(4\pi)$ since the surface of a unit sphere is 4π, (i.e., twice the surface of a hemisphere derived in Section 13.6.1). Therefore:

$$p(\theta,\phi) = \frac{\sin\theta}{4\pi}. \tag{1}$$

To show that $\boldsymbol{\omega}$ has a cosine distribution around the normal we have to show that (Section 13.6.3):

$$p(\theta',\phi) = \frac{1}{\pi}\cos\theta'\sin\theta'. \tag{2}$$

Let us look at Figure 1 (b): CE comes from $\boldsymbol{\omega_{temp}}$ so its length is 1 and angle DCE is equal to θ. This gives us that DE length is equal to $\sin\theta$. Looking at the larger right triangle ADE we see that we can also express DE length as $d\sin\theta'$. Similarly, CD length is given by $\cos\theta$ and AC comes from $\mathbf{n}$ so it has a length of 1. This means that AD length is equal to both $1+\cos\theta$ and $d\cos\theta'$. This leaves us with the following system of equations:

$$\begin{cases} d\cos\theta' = 1+\cos\theta \\ d\sin\theta' = \sin\theta \end{cases} \Leftrightarrow \begin{cases} d = \dfrac{1+\cos\theta}{\cos\theta'} \\ \tan\theta' = \dfrac{\sin\theta}{1+\cos\theta} \end{cases} \Leftrightarrow \begin{cases} d = \dfrac{1+\cos\theta}{\cos\theta'} \\ \tan\theta' = \tan\dfrac{\theta}{2} \end{cases} \tag{3}$$

$$\Leftrightarrow \quad \boxed{2\theta' = \theta}.$$

[1]McGill University: ECSE 446/556 - Realistic (& Advanced) Image Synthesis
`https://www.cim.mcgill.ca/~derek/ecse446.html`

[2]`https://psgraphics.blogspot.com/2019/02/picking-points-on-hemisphere-with.html`

Where we used the trigonometric identity $\tan\frac{x}{2} = \frac{\sin x}{1+\cos x}$. We therefore have the following transformation $(\theta,\phi) = (2\theta',\phi) \to (\theta',\phi)$. The determinant of the Jacobian is given by:

$$|J_T| = \left|\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\right| = 2. \tag{4}$$

Finally, from Section 13.6.3:

$$p(\theta',\phi) = |J_T| p(\theta,\phi). \tag{5}$$

Using Equation (1) and the trigonometric identity $\sin 2x = 2\sin x \cos x$, we can show that Equation (2) is verified and that thus the Shirley trick does indeed sample cosine distributed directions around the normal:

$$p(\theta',\phi) = |J_T| p(\theta,\phi) = 2\frac{\sin\theta}{4\pi} = \frac{\sin 2\theta'}{2\pi} = \frac{1}{\pi}\cos\theta' \sin\theta' \tag{6}$$

$\square$

References

Matt Pharr, Wenzel Jakob, and Greg Humphreys. *Physically based rendering: From theory to implementation.* MIT Press, 2023.